

Grid-Forming Inverters & Virtual Synchronous Machines

The math behind the episode: swing equation, inertia, droop, power–angle, and the VSM control law.

1. The swing equation, from rotor to per unit

For a synchronous machine with moment of inertia J , mechanical speed ω_m , and turbine and electrical torques T_m, T_e :

$$J \frac{d\omega_m}{dt} = T_m - T_e \quad \implies \quad J\omega_m \frac{d\omega_m}{dt} = P_m - P_e.$$

Define the **inertia constant** as stored kinetic energy at synchronous speed over the machine rating:

$$H = \frac{\frac{1}{2} J \omega_{sm}^2}{S_{base}} \quad [\text{s}].$$

Substituting, and using $\omega \approx \omega_s$ (frequency stays within a fraction of a percent), gives the per-unit form, written directly in electrical frequency f :

$$\frac{2H}{f_0} \frac{df}{dt} = P_m - P_e \quad (\text{per unit on } S_{base}).$$

Machines on the same grid are synchronized, so their inertias add, weighted by rating:

$$H_{sys} = \frac{\sum_i H_i S_i}{\sum_i S_i}.$$

Initial RoCoF

At the instant a power imbalance ΔP (per unit of online capacity) appears, before governors or controllers respond:

$$\text{RoCoF} = \left. \frac{df}{dt} \right|_{t=0^+} = -\frac{\Delta P f_0}{2H_{sys}}.$$

RoCoF is inversely proportional to inertia: halve H_{sys} and RoCoF doubles.

2. Droop: the governor layer

Primary control makes each unit's output a proportional function of frequency error. With droop R (typically 4–5%):

$$\frac{\Delta f}{f_0} = -R \frac{\Delta P}{P_{rated}}.$$

If every unit has the same R , the steady-state deviation after a loss ΔP (per unit of the responding fleet, ignoring load damping) is $\Delta f_{ss} = -R f_0 \Delta P$. Inertia sets how fast frequency falls; droop sets where it settles. The lowest point in between, the *nadir*, depends on both, and on how fast the governors act.

3. Power–angle and the P–f / Q–V split

Two voltage sources $V_1 \angle \delta$ and $V_2 \angle 0$ joined by a mostly inductive reactance X :

$$P = \frac{V_1 V_2}{X} \sin \delta, \quad Q = \frac{V_1 (V_1 - V_2 \cos \delta)}{X}.$$

For small δ : $P \approx V_1 V_2 \delta / X$ depends mainly on **angle**, and $Q \approx V_1 (V_1 - V_2) / X$ mainly on **voltage magnitude**. Angle is the integral of frequency, so real power is controlled through frequency and reactive power through voltage. That is why grid-forming controllers use P–f and Q–V laws. (The split weakens on resistive low-voltage networks, where R/X is not small.)

4. Grid-following vs. grid-forming, in equations

	Grid-following (GFL)	Grid-forming (GFM)
Behaves as	Controlled current source	Voltage source behind impedance
Angle comes from	PLL tracking the grid voltage	Its own control law (droop / VSM / VOC)
Commands	i_d^*, i_q^* aligned to PLL angle	$E \angle \theta$ set internally
Weak grid (low SCR)	PLL–grid interaction can destabilize	Stable; stiffens the grid

Grid strength at the connection point is measured by the **short-circuit ratio** $SCR = S_{sc} / P_{rated}$. Below about 3 is commonly called weak, and below 2 very weak. In a weak grid the GFL inverter's own current moves the voltage its PLL is measuring, and that feedback loop is what goes unstable.

5. The virtual synchronous machine control law

A VSM integrates the swing equation in firmware to produce the inverter's internal voltage angle θ . In per unit:

$$2H_v \frac{d\omega}{dt} = P_{ref} - P_e - D_p(\omega - \omega_0), \quad \frac{d\theta}{dt} = \omega_b \omega,$$

$$E = E_{ref} + k_q (Q_{ref} - Q).$$

H_v (virtual inertia) and D_p (damping) are **parameters**, not physics. Setting $D_p = 1/R$ with $H_v \rightarrow 0$ reduces the law to plain droop, so droop control is the zero-inertia special case of a VSM.

How much energy does "virtual inertia" cost?

Energy the DC side must deliver while frequency falls from f_0 to $f_0 - \Delta f$, for an inverter rated S :

$$E = H_v S \left[1 - \left(\frac{f_0 - \Delta f}{f_0} \right)^2 \right] \approx 2 H_v S \frac{\Delta f}{f_0}.$$

Example: $S = 100$ MVA, $H_v = 4$ s, $\Delta f = 0.5$ Hz at 60 Hz gives $E \approx 6.7$ MJ ≈ 1.9 kWh, which is trivial for a 100 MWh battery. The real constraint is **current**. A synchronous machine can push several per unit of fault current for a short time, while an inverter is usually limited to roughly 1.1–1.5 per unit. Grid-forming designs spend their effort on behaving well when they hit that limit.

6. Self-test, worked

Exercise 1: the inertia crisis in one calculation

Take $f_0 = 60$ Hz, $H_{\text{sys}} = 4$ s, and a sudden loss of 5% of online generation ($\Delta P = 0.05$ pu):

$$\text{RoCoF} = \frac{0.05 \times 60}{2 \times 4} = 0.375 \text{ Hz/s}.$$

Halve the inertia to $H_{\text{sys}} = 2$ s and RoCoF becomes **0.75** Hz/s. For scale, GB's distribution-connected loss-of-mains protection historically used RoCoF settings as low as 0.125 Hz/s. Those settings tripped embedded generation in the August 2019 event and were relaxed afterward. Faster RoCoF also leaves governors less time before the nadir.

Steady state, if droop is 5% across the remaining fleet: $\Delta f_{ss} \approx -0.05 \times 60 \times 0.05 = -0.15$ Hz.

Exercise 2: why followers fall over

A GFL inverter injects current at an angle given by its PLL. At low SCR the grid impedance is large, so that current noticeably shifts the point-of-connection voltage, and with it the angle the PLL is tracking. The inverter is partly measuring itself, and the loop gain rises as SCR falls until the PLL and current controller lose phase margin. A GFM inverter creates its own voltage reference, so there is no external angle to chase. A weak grid just means its voltage dominates more of the local network.